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1992 J. Phys. A: Math. Gen. 25 861

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J. Phys. A: Math. Gen. 25 (1992) 861–876. Printed in the UK

Group theoretical basis for the terminating $_3F_2(1)$ series

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Received 28 January 1991, in final form 29 July 1991

Abstract. It is shown that a recursive use of the transformation for a terminating $_3F_2(1)$ series used by Weber and Erdelyi, which belongs, as shown by Whipple, to a set of equivalent $_3F_2(1)$ functions obtained by Thomae, results in a 72-element group associated with 18 terminating series. The generators, conjugacy classes, invariant subgroups, characters and dimensions of irreducible representations for this group are presented.

1. Introduction

The terminating generalized hypergeometric functions of unit argument, the $_3F_2(1)$ and the $_4F_3(1)$, have been related to the Wigner (3-j) and the Racah (6-j) coefficients, respectively, in the literature (see, for example, Smorodinskii and Shelepin 1972; Biedenharn and Louck 1981a, b). Starting with the van der Waerden (1932) form for the 3-j coefficient, and resorting to the comprehensive work of Whipple (1925) on the symmetries of the $_3F_2(1)$ functions, Raynal (1978) obtained ten different forms for the 3-j coefficient. A set of six $_3F_2(1)$s of the van der Waerden form has been shown (Srinivasa Rao 1978) to be necessary and sufficient to account for the 72 symmetries of the 3-j coefficient. With a transformation for a terminating $_3F_2(1)$ series, used by Weber and Erdelyi (1952), Rajeswari and Srinivasa Rao (1989) derived from the van der Waerden set of six $_3F_2(1)$s, three other sets of $_3F_2(1)$s corresponding to the Wigner (1940), Racah (1942) and Majumdar (1955) forms. They also studied the consequences of relating the Majumdar form of $_3F_2(1)$ for the 3-j coefficient to the discrete orthogonal Hahn polynomial to obtain recurrence relations satisfied by the 3-j coefficient.

Recently, Beyer *et al* (1987) showed that an identity due to Thomae (1879) between two $_3F_2(1)$ series, together with invariance under separate permutations of numerator and denominator parameters, implies that the symmetric group S_5 is an invariance group of the non-terminating series. In the same paper, Bailey's transformation for the terminating Saalschützian $_4F_3$ series (Bailey 1935, p 56) is used to study the symmetry group of two-term relations for this series, which is also S_5. Using the relation between the 6-j coefficient and the terminating Saalschützian

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0305-4470/92/040861+16$04.50 © 1992 IOP Publishing Ltd

${}_4F_3(1)$, and applying this new symmetry group, it is shown (Louck *et al* 1987) that the classical group of 144 symmetries (the Regge symmetries of the 6-j coefficient) are extended to a group of 23 040 symmetries by extending the domain of these coefficients. Clearly, this extended domain contains also 'unphysical' arguments for the 6-j coefficient. Note that the extended symmetry group of order 23 040 had already been encountered by D'Adda *et al* (1972,1974) in their unified treatment of $SU(2)$ and $SU(1,1)$ 6-j coefficients.

It would be of interest to make a similar analysis for the 3-j coefficient and its representing series, the terminating ${}_3F_2(1)$. Let us point out here that the group S_5 of Beyer *et al* (1987) is the symmetry group of two-term relations for the non-terminating ${}_3F_2(1)$ function. Since it is the terminating ${}_3F_2(1)$ series which is related to 3-j coefficients, we study in this article the corresponding symmetry group of two-term relations for the same.

A transformation for a terminating ${}_3F_2(1)$ series given by Weber and Erdelyi (1952), when used recursively, is shown to generate all the 18 terminating series on which are superposed the trivial $S_2 \times S_2$ symmetry – a consequence of the invariance of a given terminating ${}_3F_2(1)$ series to the permutation of its two numerators (note that the third numerator parameter determines the termination of the series) and its two denominator parameters. This 72-element finite group of transformations has nine conjugacy classes and correspondingly nine irreducible representations (irreps) four of dimension one, one of dimension two and four of dimension four. The three generators for this 72-element group are given. The smallest invariant or normal subgroup, H_9 (say), of this finite group is of order nine, and it is imbedded in an 18 element invariant subgroup, H_{18} and H_{18}, in turn, is imbedded in three 36-element invariant subgroups. In terms of Whipple's parameters (1925) for the ${}_3F_2(1)$, it is shown that H_9 is isomorphic to the product of two cyclic groups of order 3.

The 72-element group G_{T} is shown to be the invariance group of ${}_3\tilde{F}_2$, which is a rescaling of the terminating ${}_3F_2$. Thus it is the group generating all two-term relations for this series. The phase factor appearing in such a two-term relation is shown to be equal to an irreducible character of G_{T}, motivating the construction of the complete character table for G_{T}. Using the van der Waerden form for the 3-j coefficient, the implication of the symmetry group G_{T} on 3-j symbols is investigated. The conclusion is similar to that for the 6-j symbols (Louck *et al* 1987): the classical group of 72 symmetries (the Regge symmetries of the 3-j coefficient) are extended to a group of 1440 symmetries by extension of the domain of these coefficients. Again, this extended domain contains 'unphysical' arguments.

In section 2, the essential notation required is given. In section 3, starting with a matrix representing the Weber–Erdelyi transformation for a terminating ${}_3F_2(1)$ series, the procedure for generating the 72-element group G_{T} is described and the Whipple parametrization introduced. In section 4, the structure of the group G_{T}, its conjugacy classes, its irreps and their corresponding characters, and the invariant subgroups of G_{T} are presented. In section 5, comments and conclusions regarding a scaling transformation which makes G_{T} an invariance group of the terminating ${}_3F_2(1)$ series, the use of the symmetry group in the context of the 3-j coefficient, etc. are made. Finally, in an appendix, the 18 transformations of the ${}_3F_2(1)$ are stated explicitly, in the Whipple notation and in a scaled, invariant form.

2. Notation

Whipple (1925) introduced six parameters r_i, $i = 0,1,2,3,4,5$, such that

$$\sum_{i=0}^{5} r_i = 0 \tag{1}$$

and let

$$\alpha_{lmn} = \tfrac{1}{2} + r_l + r_m + r_n \qquad \beta_{mn} = 1 + r_m - r_n. \tag{2}$$

With these he defined the function:

$$F_p(l;mn) = \frac{1}{\Gamma(\alpha_{ijk}, \beta_{ml}, \beta_{nl})}\, {}_3F_2\begin{pmatrix}\alpha_{imn}, \alpha_{jmn}, \alpha_{kmn}; 1\\ \beta_{ml}, \beta_{nl}\end{pmatrix} \tag{3}$$

where i, j and k are used to represent those three numbers out of the six integers 0,1,2,3,4,5 not already represented by l, m and n. The function ${}_3F_2(1)$ is the generalized hypergeometric function (cf Slater 1966) of unit argument having $\alpha_{imn}, \alpha_{jmn}, \alpha_{kmn}$ as its three numerator parameters and β_{ml}, β_{nl} as its two denominator parameters. By changing the signs of all the r_i parameters and using the constraint (1), Whipple defined another function:

$$F_n(l;mn) = \frac{1}{\Gamma(\alpha_{lmn}, \beta_{lm}, \beta_{ln})}\, {}_3F_2\begin{pmatrix}\alpha_{ljk}, \alpha_{lik}, \alpha_{lij}; 1\\ \beta_{lm}, \beta_{ln}\end{pmatrix}. \tag{4}$$

In (3) and (4) use is made of the notation:

$$\Gamma(x, y, z, \ldots) = \Gamma(x)\Gamma(y)\Gamma(z)\ldots. \tag{5}$$

By permutation of the suffixes l, m, n over the six integers 0,1,2,3,4,5, then 60 F_p functions and 60 F_n functions can be written down. If there is no negative integer in the numerator parameters, these series converge only if the real parts of α_{ijk} in (3) and α_{lmn} in (4) are positive. For the sake of brevity the unit argument of the generalized hypergeometric series will not be displayed and it will be denoted as ${}_3F_2(\binom{a,b,c}{d,e})$ or ${}_3F_2(a,b,c;d,e)$, the three numerator and the two denominator parameters being the variables.
(*Note*: The use of n as a suffix for the F_n function and also as an index for α and β is continued here as in the literature.)

3. Terminating series

Consider the transformation for a terminating ${}_3F_2$ used by Weber and Erdelyi (1952):

$${}_3F_2\begin{pmatrix}a, b, -N\\ d, e\end{pmatrix} = \frac{\Gamma(d, d+N-a)}{\Gamma(d+N, d-a)}\, {}_3F_2\begin{pmatrix}a, e-b, -N\\ 1+a-d-N, e\end{pmatrix}. \tag{6}$$

This formula is one of a set (cf Bailey 1935) obtained by Whipple (1925). If the five parameters of the ${}_3F_2$ on the LHS of (7) are denoted by the column vector:

$$\boldsymbol{x} = (a, b, 1-N, d, e) \tag{7}$$

then the parameters of the ${}_3F_2$ on the RHS of (7) are obtained when the matrix:

$$g_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

operates on $\boldsymbol{x}$. Note that $1-N$ is used instead of $-N$, as a component of the column vector $\boldsymbol{x}$, since it represents the number of terms in a terminating series. However, ${}_3F_2(a,b,-N;d,e)$ will be denoted by ${}_3F_2(\boldsymbol{x})$.

Using (6) again, with the roles of d and e interchanged, to transform the RHS of (6), Weber and Erdelyi obtained the transformation:

$$\begin{aligned} {}_3F_2\left(\begin{matrix} a,b,-N \\ d,e \end{matrix}\right) &= \frac{\Gamma(d,e,e+N-a,d+N-a)}{\Gamma(d+N,e+N,d-a,e-a)} \\ &\quad \times {}_3F_2\left(\begin{matrix} a,1-s,-N \\ 1-b+d-s,1-b+e-s \end{matrix}\right) \end{aligned} \tag{9}$$

where $s = d+e-a-b+N$. The question arises as to whether this recursive use of the Weber–Erdelyi transformation (6) can be continued. In fact, such a procedure when continued results in a group of 72 transformations, which are the 18 terminating ${}_3F_2$ series (see appendix) on which are superposed the $a \leftrightarrow b$, $d \leftrightarrow e$ and ($a \leftrightarrow b$, $d \leftrightarrow e$) interchanges.

Let g_2 be the matrix

$$g_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

which interchanges a and b when it operates on $\boldsymbol{x}$ and denote by g_3 the matrix:

$$g_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{11}$$

which interchanges d and e when it operates on $\boldsymbol{x}$. By forming all possible products of all possible powers of g_1, g_2 and g_3, a group of 72 transformation matrices can be generated which provides a 5×5 representation for the terminating series, with (7) as the basis. Thus, g_1, g_2 and g_3 are the generators of a group G_{T} for the transformations of a terminating ${}_3F_2$ series, with $g_i^2 = \mathbf{1}$, for $i = 1,2,3$.

A similarity transformation, $u^{-1}g_i u$, with:

$$u = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad u^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & -2 & 0 & 3 \end{bmatrix} \tag{12}$$

block diagonalizes the generators, and hence all the $g \in G_{\mathrm{T}}$, thereby reducing the generators for the 5×5 representation into the generators for a one-dimensional identity irrep (due to $-N$ being kept fixed in (6)) and the generators for a four-dimensional faithful irrep given by:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{13}$$

In terms of Whipple's parameters and the definitions for F_p and F_n series given by (3) and (4), respectively, the transformation (6) can be written as:

$$F_p(0;45) = (-1)^N \frac{\Gamma(\alpha_{015}, \alpha_{025})}{\Gamma(\alpha_{123}, \alpha_{124})} F_n(5;02) \tag{14}$$

where $\alpha_{345} = -N$. (See appendix and equation (4.3.3.6) in Slater 1966.) In the Whipple parameter basis, where

$$x' = (r_0, r_1, r_2, r_3, r_4, r_5) \tag{15}$$

is represented as a column vector, the transformation (14) is equivalent to the 6×6 transformation matrix:

$$g_1' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{16}$$

The permutation of the two numerator parameters a and b in the ${}_3F_2$, in terms of Whipple parameters is equivalent to an interchange of r_1 and r_2, which is induced by the matrix:

$$g_2' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

operating on the basis vector x'. Similarly, the permutation of the two denominator parameters d and e in the ${}_3F_2$, is equivalent to the interchange of r_4 and r_5, induced by:

$$g_3' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{18}$$

These three 6×6 matrices generate a six-dimensional reducible representation for G_T.

This six-dimensional representation, in the Whipple parameter basis, x', can be reduced by the similarity transformation, $u'^{-1} g'_i u'$, with:

$$u' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$u'^{-1} = \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & -2 & 0 & 0 & 0 \\ 2 & 2 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -2 & -2 \\ 0 & 0 & 0 & 2 & 2 & -4 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \tag{19}$$

which block diagonalizes the generators g'_1, g'_2 and g'_3, and hence all the $g' \in G_T$. It results in two one-dimensional irreps, one of which is the identity irrep, and a four-dimensional faithful irrep with generators:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \tag{20}$$

From (16)–(18) it follows that G_T is a subgroup of the permutation group S_6. Indeed, the generators g'_i of G_T can be represented by 6×6 permutation matrices (including an overall minus-sign for g'_1). If we use the cycle notation for an element of S_6 represented by a 6×6 permutation matrix, we see from (16)–(18) that

$$\begin{aligned} g'_1 &= -(05)(13)(24) \\ g'_2 &= (12) \\ g'_3 &= (45) \end{aligned} \tag{21}$$

where a minus sign for g'_1 is included in order to remember that in the Whipple parameter representation this generator is actually a permutation matrix multiplied by -1. In the following section it will be very useful to represent elements of G_T by means of the above cycle notation, especially for distinguishing between conjugacy classes with the same order.

4. Structure of G_T and its irreps

Two elements h and h' of a group G are said to be conjugate if there exists a $g \in G$ such that $h' = ghg^{-1}$. This defines an equivalence relation on G, the equivalence classes being called the conjugacy classes. Analysis of G_T, reveals that there are nine conjugacy classes $K_1, \ldots, K_9$. A conjugacy class is represented by

Table 1.

class	order k_i of K_i	order of $g \in K_i$	representative of $g \in K_i$	
K_1	1	0	$g_1^2 = 1$	1
K_2	4	3	$g_1 g_2 g_1 g_3$	(345)
K_3	4	3	$(g_2 g_1 g_3)^2$	(012)(345)
K_4	6	2	g_1	−(05)(13)(24)
K_5	6	2	g_2	(12)
K_6	9	2	$g_2 g_3$	(12)(45)
K_7	12	6	$g_2 g_1 g_3$	−(051324)
K_8	12	6	$g_1 g_3 g_2 g_1 g_2$	(021)(34)
K_9	18	4	$g_1 g_2$	−(05)(1423)

one of its elements. In table 1 are given the list of all the conjugacy classes K_i, a representative element (given in terms of the generators, and as a permutation matrix in cycle notation), the order k_i of K_i (i.e. the number of elements of K_i), and the order of the elements of K_i (i.e. the smallest integer s such that $g^s = 1$, for $g \in K_i$).

Following the general theory of group representations (cf Wybourne 1970 or Messiah 1964), the table of characters for the irreps of G_T has been obtained. As there are nine conjugacy classes, there are nine inequivalent irreps, which are denoted by $D^{(1)}, \ldots, D^{(9)}$. Four irreps are of dimension one, one is of dimension two, and four are of dimension four. It is only the four-dimensional irreps which are faithful. Table 2 lists the characters.

Table 2.

	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
$D^{(1)}$	1	1	1	1	1	1	1	1	1
$D^{(2)}$	1	1	1	−1	1	1	−1	1	−1
$D^{(3)}$	1	1	1	1	−1	1	1	−1	−1
$D^{(4)}$	1	1	1	−1	−1	1	−1	−1	1
$D^{(5)}$	2	2	2	0	0	−2	0	0	0
$D^{(6)}$	4	1	−2	0	2	0	0	−1	0
$D^{(7)}$	4	1	−2	0	−2	0	0	1	0
$D^{(8)}$	4	−2	1	2	0	0	−1	0	0
$D^{(9)}$	4	−2	1	−2	0	0	1	0	0

Simply by looking at the traces of g_1 and g_2, and comparing with the columns K_4 and K_5 (of which g_1 and g_2 are representatives) in the character table 2, it is possible to conclude that the representation generated by g_i $(i = 1, 2, 3)$ is equivalent to

$$D^{(1)} \oplus D^{(6)}$$

and that the Whipple parameter representation generated by $g_i'(i = 1, 2, 3)$ is equivalent to

$$D^{(1)} \oplus D^{(2)} \oplus D^{(6)}.$$

As a consequence, the irreducible representation matrices (13) and (20) for the generators of G_T are equivalent and both can be labelled by $D^{(6)}$.

The next property to analyse is the simplicity of G_{T}. All the invariant subgroups H of G_{T} have been found. Among these there are proper Abelian invariant subgroups, hence G_{T} is neither simple nor semi-simple. Recall that a subgroup H is an invariant subgroup (self-conjugate subgroup, normal divisor) if $G_T H G_T^{-1} = H$. To find invariant subgroups, one can form unions of conjugacy classes and check if they close under the group multiplication law. The following inclusions give a complete list of the invariant subgroups of G_{T} (the subscript denoting the order of H):

$$H_9 \subset H_{18} \begin{cases} \subset H_{36} \subset G_T \\ \subset H'_{36} \subset G_T \\ \subset H''_{36} \subset G_T \end{cases} \tag{22}$$

where

$$\begin{aligned} H_9 &= K_1 \cup K_2 \cup K_3 \\ H_{18} &= H_9 \cup K_6 \\ H_{36} &= H_{18} \cup K_9 \\ H'_{36} &= H_{18} \cup K_4 \cup K_7 \\ H''_{36} &= H_{18} \cup K_5 \cup K_8. \end{aligned} \tag{23}$$

It should be noted that, in terms of the three generators g_i (or g'_i) introduced previously, one can write

$$\begin{aligned} &K_6 = g_2 g_3 H_9 && K_9 = g_1 g_2 H_{18} \\ &K_4 \cup K_7 = g_1 H_{18} && K_5 \cup K_8 = g_2 H_{18} \end{aligned} \tag{24}$$

such that the invariant subgroups (23) can be characterized as follows in terms of H_9 and the three generators:

$$\begin{aligned} H_{18} &= H_9 \cup g_2 g_3 H_9 \\ H_{36} &= H_9 \cup g_2 g_3 H_9 \cup g_1 g_2 H_9 \cup g_1 g_3 H_9 \\ H'_{36} &= H_9 \cup g_2 g_3 H_9 \cup g_1 H_9 \cup g_1 g_2 g_3 H_9 \\ H''_{36} &= H_9 \cup g_2 g_3 H_9 \cup g_2 H_9 \cup g_3 H_9. \end{aligned} \tag{25}$$

The smallest invariant subgroup, H_9, is easy to characterize. In fact $H_9 = C_3 \times C_3$, the direct product of two cyclic groups on three elements. In terms of the Whipple parametrization, the generators of the two C_3's are (012) and (345). It is now obvious that H_9 is an Abelian invariant subgroup of G_{T}.

It should be noticed that all the invariant subgroups of G_{T} can be found using the character table and the fact that those elements h of G_{T} with $\phi(h) = \phi(1)$, where ϕ is a (not necessarily simple) character of G_{T}, form an invariant subgroup (Ledermann 1977, theorem 2.7).

Conversely, having the list of all invariant subgroups of G_{T}, one can reconstruct the character table. Indeed, the first character $\chi^{(1)}$ is trivial. Next, if N is one of H_{36},

H'_{36} or H''_{36}, G/N is the two element group C_2, with non-trivial simple character $(1,-1)$. Using the 'lifting process' (Ledermann 1977, theorem 2.6), one obtains the simple characters $\chi^{(2)}$, $\chi^{(3)}$ and $\chi^{(4)}$ from H''_{36}, H'_{36} and H_{36} respectively. This completes the list of simple characters with $\chi_1^{(i)} = 1$. In order to find the remaining simple characters, the theory of induced characters can be used. If H is a subgroup of G for which a character ${}^H\phi$ is known, then

$$ {}^G\phi_i = \frac{m}{k_i} \sum_w {}^H\phi(w) \qquad w \in K_i \cap H $$

is a character (simple or compound) of G. Herein, m is the index of H and k_i is the order of K_i. As the simple characters of an Abelian group are well known, H is chosen to be $H_9 = C_3 \times C_3$, thus $m = 72/9 = 8$. Using the trivial character of H, ${}^H\phi^{(1)} = (1,1,1,1,1,1,1,1,1)$, one finds ${}^G\phi^{(1)} = (8,8,8,0,0,0,0,0,0)$. By means of the inner product for characters of G_{T},

$$ \langle \phi | \psi \rangle = \frac{1}{72} \sum_{i=1}^{9} k_i \phi_i \psi_i $$

it is found that $\langle {}^G\phi^{(1)} | \chi^{(1)} \rangle = \langle {}^G\phi^{(1)} | \chi^{(2)} \rangle = \langle {}^G\phi^{(1)} | \chi^{(3)} \rangle = \langle {}^G\phi^{(1)} | \chi^{(4)} \rangle = 1$. Thus, subtracting $\chi^{(1)}, \ldots, \chi^{(4)}$ from ${}^G\phi^{(1)}$, one obtains ${}^G\phi' = (4,4,4,0,0,-4,0,0,0)$. Since all one-dimensional irreps have been found and $\langle {}^G\phi' | {}^G\phi' \rangle = 4$, it follows that ${}^G\phi'$ is twice a simple character, i.e. ${}^G\phi' = 2\chi^{(5)}$. The next simple character, $\chi^{(6)}$, is immediately deduced from our defining representation (8), (10) and (11). Using a non-trivial character of H, ${}^H\phi^{(2)} = (1,1,1,\omega,\omega,\omega,\omega^2,\omega^2,\omega^2)$, where $\omega^2 + \omega + 1 = 0$, the inducing process leads to ${}^G\phi^{(2)} = (8,2,-4,0,0,0,0,0,0)$. One can verify that the inner product of ${}^G\phi^{(2)}$ with $\chi^{(1)}$, $\chi^{(2)}$, $\chi^{(3)}$, $\chi^{(4)}$ and $\chi^{(5)}$ is zero, and that $\langle {}^G\phi^{(2)} | \chi^{(6)} \rangle = 1$. Subtracting $\chi^{(6)}$ from ${}^G\phi^{(2)}$, one obtains ${}^G\phi'' = (4,1,-2,0,-2,0,0,1,0)$. Since $\langle {}^G\phi'' | {}^G\phi'' \rangle = 1$, it is a simple character, i.e. ${}^G\phi'' = \chi^{(7)}$. Two more simple characters $\chi^{(8)}$ and $\chi^{(9)}$ need to be found. Using the orthogonality property satisfied by the columns of the character table of G_{T}, namely

$$ \sum_{l=1}^{9} \chi_i^{(l)} \chi_j^{(l)} = \frac{72}{k_i} \delta_{ij} $$

it is a straightforward exercise to complete the character table.

5. Comments and conclusions

Although in the preceeding sections G_{T} was generated by three generators, namely the Weber–Erdelyi transformation g_1 and the two interchange transformations $a \leftrightarrow b$ (g_2) and $d \leftrightarrow e$ (g_3) it should be noted that G_{T} can actually be generated by only two elements. For instance, using the cycle structure notation for the elements of G_{T}, the 72-element group G_{T} is generated by (12) and $-(0524)(31)$, i.e. by g_2 and $(g_1 g_3)$. In fact there are many other examples of pairs of generators for G_{T}.

Using the notation of section 1, the Weber–Erdelyi transformation (6) can be written in the following form:

$$ {}_3F_2(\boldsymbol{x}) = \frac{\Gamma(d, d+N-a)}{\Gamma(d+N, d-a)}\, {}_3F_2(g_1\boldsymbol{x}) \tag{26} $$

whereas the interchange transformations are:

$$ {}_3F_2(\boldsymbol{x}) = {}_3F_2(g_2\boldsymbol{x}) \qquad {}_3F_2(\boldsymbol{x}) = {}_3F_2(g_3\boldsymbol{x}). \tag{27} $$

In general, this analysis implies that

$$ {}_3F_2(\boldsymbol{x}) = (\text{factor})\, {}_3F_2(g\boldsymbol{x}) \qquad \forall g \in G_{\mathrm{T}} \tag{28} $$

where this factor is in terms of Γ-functions, as in (6) or (9). It would be interesting if this factor could actually be determined in terms of the group element g. This can indeed be done. The most elegant way to obtain this is to perform a scaling on the ${}_3F_2(\boldsymbol{x})$:

$$ {}_3\tilde{F}_2(\boldsymbol{x}) = \frac{\Gamma(d+N, e+N)}{\Gamma(d, e)}\, {}_3F_2(\boldsymbol{x}). \tag{29} $$

Then the three generating transformations become:

$$ \begin{aligned} {}_3\tilde{F}_2(\boldsymbol{x}) &= (-1)^N\, {}_3\tilde{F}_2(g_1\boldsymbol{x}) \\ {}_3\tilde{F}_2(\boldsymbol{x}) &= {}_3\tilde{F}_2(g_2\boldsymbol{x}) = {}_3\tilde{F}_2(g_3\boldsymbol{x}). \end{aligned} \tag{30} $$

As G_{T} is generated by g_1, g_2 and g_3, the following result holds: the scaled terminating ${}_3\tilde{F}_2$ with unit argument satisfies

$$ {}_3\tilde{F}_2(\boldsymbol{x}) = {}_3\tilde{F}_2(g\boldsymbol{x}) \qquad \forall g \in G_T \qquad (\text{for } N \text{ even}) \tag{31} $$

$$ {}_3\tilde{F}_2(\boldsymbol{x}) = \chi^{(2)}(g)\, {}_3\tilde{F}_2(g\boldsymbol{x}) \qquad \forall g \in G_T \qquad (\text{for } N \text{ odd}) \tag{32} $$

where $\chi^{(2)}(g)$ is the character of g in the irrep $D^{(2)}$ (see section 4). Hence the 72-element group G_{T} can be seen as the invariance group of the terminating ${}_3F_2$. If N is odd, then the coefficient in (32) is $+1$ or -1, and it is equal to -1 if one of the following equivalent conditions is satisfied:

- g_1 appears an odd number of times in the expression of g in terms of g_1, g_2 and g_3;
- g is a permutation matrix times -1 when represented in the Whipple parametrization;
- the left and right hand sides of (32) correspond to a F_p and a F_n in terms of the notation of section 2.

The use of the Weber–Erdelyi transformation (6) on the van der Waerden ${}_3F_2$ form for the 3-j coefficient $\left(\begin{smallmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} a & b & c \\ \alpha & \beta & \gamma \end{smallmatrix}\right)$ was shown by Rajeswari and Srinivasa Rao (1989) to result in the Majumdar, Racah or Wigner ${}_3F_2$ forms, with or without the superposition of a column permutation and the $m_i \rightarrow -m_i$ substitution on them. If use is made of any one of the other transformations explicitly listed in the appendix, on the van der Waerden ${}_3F_2$ form for the 3-j coefficient, then it can be shown that the result would be one of the 12 terminating ${}_3F_2$ forms given in Raynal (1978) (namely, equations (6), (15)–(17), (26)–(30) and three others which differ from (15)–(17) by exchange of a and b and change of sign for α, β, γ in Raynal (1978), which include the Majumdar, Racah, Wigner forms) or, one of the 12 forms on which is superposed a 'classical' symmetry of the 3-j coefficient (namely, permutations of the columns of the 3-j coefficient and the $m_i \to -m_i$ substitution).

It is well known that one of the van der Waerden forms for the 3-j coefficient can be written as follows:

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \delta(m_1+m_2+m_3,0)(-1)^{j_1-j_2-m_3} \\ &\times [(-j_1+j_2+j_3)!(j_1-j_2+j_3)!(j_2-m_2)!(j_3-m_3)! \\ &\times (j_1+m_1)!(j_3+m_3)!]^{1/2} \\ &\times [(j_1+j_2-j_3)!(j_1+j_2+j_3+1)!(j_1-m_1)!(j_2+m_2)!]^{-1/2} \\ &\times [(j_3-j_1-m_2)!(j_3-j_2+m_1)!]^{-1} \\ &\times {}_3F_2\begin{pmatrix} -j_1+m_1, & -j_2-m_2, & -j_1-j_2+j_3 \\ 1+j_3-j_1-m_2, & 1+j_3-j_2+m_1 & \end{pmatrix}; 1\Bigg). \end{aligned} \tag{33}$$

Using (33), the three generating elements g_1, g_2 and g_3 of G_{T} lead, respectively, to the following symmetries of the 3-j symbol (apart from a phase factor):

$$\begin{pmatrix} j_1 & -j_3-1 & -j_2-1 \\ m_1 & m_3 & m_2 \end{pmatrix} \tag{34}$$

$$\begin{pmatrix} (j_1+j_2-m_3)/2 & (j_1+j_2+m_3)/2 & j_3 \\ (j_1-j_2+m_1-m_2)/2 & (j_1-j_2-m_1+m_2)/2 & -j_1+j_2 \end{pmatrix} \tag{35}$$

$$\begin{pmatrix} (j_1+j_2+m_3)/2 & (j_1+j_2-m_3)/2 & j_3 \\ (-j_1+j_2+m_1-m_2)/2 & (-j_1+j_2-m_1+m_2)/2 & j_1-j_2 \end{pmatrix} \tag{36}$$

The second and third of these are well known Regge symmetries of the 3-j symbol, while the first has unphysical arguments (the j-values being negative; the triangular condition is violated). The classical symmetry group of the 3-j coefficient contains 72 symmetries, of which (35) and (36) are two elements. Following Louck *et al* (1987), who extended the classical Regge group of 144 symmetries of the 6-j symbol by the ${}_4F_3$ invariance group S_5 in order to obtain a new symmetry group of order 23 040, one can perform the same process here and extend the 72 classical symmetries of the 3-j symbol by the symmetries induced by the 72-element group G_{T}. Since (35) and (36) are Regge symmetries, already contained in the 72 symmetries, this amounts to enlarging these symmetries by the element (34) and to investigating which group G it generates. In particular, (34) contains unphysical transformations of the type $j \to -j-1$ (preserving the angular momentum eigenvalue $j(j+1)$), known as Yutsis mirror symmetries (Yutsis and Bandzaitis 1965). Let us denote $j_1 \to -j_1-1$ by r'. It can be shown by recursively using r' and the column permutations of the 3-j coefficient that (34) can be transformed into

$$\begin{pmatrix} -j_1-1 & j_2 & j_3 \\ m_1 & m_2 & m_3. \end{pmatrix} \tag{37}$$

The group G can be generated by the classical symmetries together with r'. This new group G is of order 1440; it can be interpreted as the extended symmetry group of the 3-j coefficient by extending the domain of this coefficient. This extended domain contains unphysical arguments. It should be noticed that this extended symmetry group of order 1440 has been encountered by D'Adda *et al* (1972), in treating $SU(2)$ and $SU(1,1)$ 3-j coefficients, and by Huszár (1972). There are two further observations to make. The first is that the 'trivial' ${}_3F_2$ symmetry permuting two of the three numerator parameters corresponds to a non-trivial Regge symmetry for the 3-j symbol (in fact, this observation is not new: see Biedenharn and Louck (1981b), p 433). The second, new, observation is that a 'trivial' 3-j symmetry (namely $j_1 \to -j_1 - 1$) corresponds to a non-trivial transformation for the terminating ${}_3F_2(1)$ series, namely to (6).

It is considered relevant to point out the contemporary work of Beyer *et al* (1987) in the present context. For this purpose, in the Whipple notation (section 2) let l, m, n be 0, 4, 5, respectively. Then the numerator and denominator parameters which occur in $F_p(0;45)$, given by (3), after elimination of r_0 using (1), are related to the five independent Whipple parameters:

$$\boldsymbol{r} = (r_1, r_2, r_3, r_4, r_5) \tag{38}$$

through the transformation:

$$\boldsymbol{\alpha} = A\boldsymbol{r} \tag{39}$$

where

$$\boldsymbol{\alpha} = (\alpha_{145} - \tfrac{1}{2}, \alpha_{245} - \tfrac{1}{2}, \alpha_{345} - \tfrac{1}{2}, \beta_{40} - 1, \beta_{50} - 1)$$

and

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}. \tag{40}$$

This 5×5 matrix A plays a crucial role in the study of the group structure of two-term identities by Beyer *et al* (1987). They analyse the group structure of the non-terminating series and establish that the symmetric group S_5 is an invariance group of the two-term relation for the ${}_3F_2$ series due to Thomae (1879) and the invariance of that series to separate permutations of the numerator and denominator parameters of the ${}_3F_2$.

In this article, we generated a 72-element group G_{T} for the terminating ${}_3F_2(1)$ series, presented the conjugacy classes, irreps and their characters, and the invariant subgroups of G_{T} and discussed the role of these terminating series for the ${}_3F_2(1)$ forms of the 3-j coefficient.

The group G_{T}, of interest for us has been arrived at by a simple recursive use of a given ${}_3F_3(1)$ transformation and the results presented for the terminating ${}_3F_2(1)$ series supplement the work of Beyer *et al* (1985). The structure of the invariance group G_{T} for the terminating ${}_3F_2(1)$ series has turned out to be more intricate than that of the symmetric group S_5 shown to be the invariance group for the non-terminating ${}_3F_2(1)$ series investigated by Beyer *et al* (1985). Our study contributes to a complete understanding of an interesting aspect overlooked in the work of Beyer *et al* (1985).

Acknowledgments

One of us (KSR) wishes to thank Professor Guido Vanden Berghe and the NFWO Belgium for hospitality at the Rijksuniversiteit Gent and for financial support, and another (VR) acknowledges with thanks the award of a Research Associateship by the University Grants Commission of India. The authors thank Dr Hans De Meyer for a careful reading of the manuscript and Ms P Mortier for her diligent typing of the same in LaTeX.

Appendix

In this appendix, the 18 terminating ${}_3F_2$ transformations are written down explicitly as they arise when the Weber–Erdelyi transformation (6) is recursively used. They are expressed then in terms of Whipple parametrization and finally using the scaling transformation which enabled us to show G_{T} as the invariance group of the terminating ${}_3F_2$.

$$
{}_3F_2\begin{pmatrix} a, b, -N \\ d, e \end{pmatrix} = \frac{(d-a,N)}{(d,N)}\,{}_3F_2\begin{pmatrix} a, e-b, -N \\ 1+a-d-N, e \end{pmatrix} \tag{I}
$$

$$
= (-1)^N \frac{(1-s,N)}{(d,N)}\,{}_3F_2\begin{pmatrix} e-a, e-b, -N \\ s-N, e \end{pmatrix} \tag{II}
$$

$$
= \frac{(d-a,N)(e-a,N)}{(d,N)(e,N)}\,{}_3F_2\begin{pmatrix} a, 1-s, -N \\ 1+a-d-N, 1+a-e-N \end{pmatrix} \tag{III}
$$

$$
= \frac{(d-a,N)(b,N)}{(d,N)(e,N)}\,{}_3F_2\begin{pmatrix} e-b, 1-d-N, -N \\ 1-b-N, 1+a-d-N \end{pmatrix} \tag{IV}
$$

$$
= \frac{(d-b,N)}{(d,N)}\,{}_3F_2\begin{pmatrix} e-a, b, -N \\ 1+b-d-N, e \end{pmatrix} \tag{V}
$$

$$
= (-1)^N \frac{(1-s,N)(b,N)}{(d,N)(e,N)}\,{}_3F_2\begin{pmatrix} e-b, d-b, -N \\ 1-b-N, s-N \end{pmatrix} \tag{VI}
$$

$$
= (-1)^N \frac{(1-s,N)(a,N)}{(d,N)(e,N)}\,{}_3F_2\begin{pmatrix} e-a, d-a, -N \\ 1-a-N, s-N \end{pmatrix} \tag{VII}
$$

$$
= (-1)^N \frac{(d-a,N)(d-b,N)}{(d,N)(e,N)}\,{}_3F_2\begin{pmatrix} 1-s, 1-d-N, -N \\ 1+a-d-N, 1+b-d-N \end{pmatrix} \tag{VIII}
$$

$$
= (-1)^N \frac{(e-a,N)}{(e,N)}\,{}_3F_2\begin{pmatrix} a, d-b, -N \\ d, 1+a-e-N \end{pmatrix} \tag{IX}
$$

$$
= (-1)^N \frac{(e-a,N)(e-b,N)}{(d,N)(e,N)}\,{}_3F_2\begin{pmatrix} 1-s, 1-e-N, -N \\ 1+a-e-N, 1+b-e-N \end{pmatrix} \tag{X}
$$

$$
= (-1)^N \frac{(a,N)(b,N)}{(d,N)(e,N)}\,{}_3F_2\begin{pmatrix} 1-d-N, 1-e-N, -N \\ 1-a-N, 1-b-N \end{pmatrix} \tag{XI}
$$

$$
= {}_3F_2\begin{pmatrix} a, b, -N \\ d, e \end{pmatrix} \qquad \text{(identity)} \tag{XII}
$$

$$= \frac{(d-b,N)(a,N)}{(d,N)(e,N)} {}_3F_2\begin{pmatrix} e-a, 1-d-N, -N \\ 1-a-N, 1+b-d-N \end{pmatrix} \tag{XIII}$$

$$= \frac{(d-b,N)(e-b,N)}{(d,N)(e,N)} {}_3F_2\begin{pmatrix} b, 1-s, -N \\ 1+b-d-N, 1+b-e-N \end{pmatrix} \tag{XIV}$$

$$= \frac{(1-s,N)}{(e,N)} {}_3F_2\begin{pmatrix} d-a, d-b, -N \\ d, s-N \end{pmatrix} \tag{XV}$$

$$= \frac{(b,N)(e-a,N)}{(d,N)(e,N)} {}_3F_2\begin{pmatrix} d-b, 1-e-N, -N \\ 1+a-e-N, 1-b-N \end{pmatrix} \tag{XVI}$$

$$= \frac{(a,N)(e-b,N)}{(d,N)(e,N)} {}_3F_2\begin{pmatrix} d-a, 1-e-N, -N \\ 1-a-N, 1+b-e-N \end{pmatrix} \tag{XVII}$$

$$= \frac{(e-b,N)}{(e,N)} {}_3F_2\begin{pmatrix} b, d-a, -N \\ d, 1+b-e-N \end{pmatrix} \tag{XVIII}$$

where $s = d+e-a-b+N$ and $(\alpha, N) = \Gamma(\alpha+N)/\Gamma(\alpha)$. These transformations reduce to five relations when they are written in terms of Whipple parameters and the notation of Whipple given in section 2. They are:

$$\Gamma(\alpha_{123}, \alpha_{124}, \alpha_{125}) F_p(0) = \Gamma(\alpha_{023}, \alpha_{024}, \alpha_{025}) F_p(1) \tag{A.1}$$

$$= \Gamma(\alpha_{013}, \alpha_{014}, \alpha_{015}) F_p(2) \tag{A.2}$$

$$= (-1)^N \Gamma(\alpha_{123}, \alpha_{013}, \alpha_{023}) F_n(3) \tag{A.3}$$

$$= (-1)^N \Gamma(\alpha_{124}, \alpha_{014}, \alpha_{024}) F_n(4) \tag{A.4}$$

$$= (-1)^N \Gamma(\alpha_{125}, \alpha_{015}, \alpha_{025}) F_n(5) \tag{A.5}$$

where

(A.1) represents (XIII), (XIV) and (XVII)
(A.2) represents (III), (IV) and (XVI)
(A.3) represents (VI), (VII) and (XI)
(A.4) represents (IX), (X) and (XVIII)
(A.5) represents (I), (V) and (VIII)

while (XII) is the identity; (II) and (XV) correspond to $F_p(0;45) = F_p(0;35)$ and $F_p(0;45) = F_p(0;34)$, respectively. These relations: $F_p(0;45) = F_p(0;35) = F_p(0;34)$ represent the fact that for a given l, all the ten expressions $F_p(l;mn)$ (as well as, all the ten $F_n(l;mn)$) are equal. It is for this reason that they are denoted simply as $F_p(l)$ or $F_n(l)$ above. The relations (A.1) to (A.5) are the same as (4.3.3.2) to (4.3.3.6) in Slater (1966), who has also tabulated the expressions for α (and β) in terms of $a, b, c(= -N), d, e$ (cf table 4.1 of Slater 1966). The transformation (XI) represents the reversal of series.

If the scaling transformation (29) is used in the definitions (3) and (4) for the $F_p(l;mn)$ and $F_n(l;mn)$ functions, then for $\alpha_{kmn} = -N$:

$$F_p(l;mn) = \frac{1}{\Gamma(\alpha_{ijk}, \alpha_{ijm}, \alpha_{ijn})} {}_3\tilde{F}_2\begin{pmatrix} \alpha_{imn}, \alpha_{jmn}, -N \\ \beta_{ml}, \beta_{nl} \end{pmatrix} \tag{A.6}$$

and

$$F_n(l;mn) = \frac{1}{\Gamma(\alpha_{lmn}, \alpha_{kln}, \alpha_{klm})} {}_3\tilde{F}_2\begin{pmatrix} \alpha_{ljk}, \alpha_{lik}, -N \\ \beta_{lm}, \beta_{ln} \end{pmatrix}. \tag{A.7}$$

Redefining:

$$\tilde{F}_p(l;mn) = \Gamma(\alpha_{ijk}, \alpha_{ijm}, \alpha_{ijn}) F_p(l;mn) \tag{A.8}$$

and

$$\tilde{F}_n(l;mn) = \Gamma(\alpha_{lmn}, \alpha_{kln}, \alpha_{klm}) F_n(l;mn) \tag{A.9}$$

for $\alpha_{345} = -N$, the relations (A.1) to (A.5) will now become simply:

$$\begin{aligned} \tilde{F}_p(0) &= \tilde{F}_p(1) = \tilde{F}_p(2) \\ &= (-1)^N \tilde{F}_n(3) = (-1)^N \tilde{F}_n(4) = (-1)^N \tilde{F}_n(5) \end{aligned} \tag{A.10}$$

since

$$\begin{aligned} \tilde{F}_p(0) &= \Gamma(\alpha_{123}, \alpha_{124}, \alpha_{125}) F_p(0) \\ \tilde{F}_p(1) &= \Gamma(\alpha_{023}, \alpha_{024}, \alpha_{025}) F_p(1) \\ \tilde{F}_p(2) &= \Gamma(\alpha_{013}, \alpha_{014}, \alpha_{015}) F_p(2) \\ \tilde{F}_p(3) &= \Gamma(\alpha_{123}, \alpha_{013}, \alpha_{023}) F_n(3) \\ \tilde{F}_p(4) &= \Gamma(\alpha_{124}, \alpha_{014}, \alpha_{024}) F_n(4) \\ \tilde{F}_p(5) &= \Gamma(\alpha_{125}, \alpha_{015}, \alpha_{025}) F_n(5). \end{aligned} \tag{A.11}$$

In general, for any $\alpha_{lmn} = -N$, the relations among the 18 terminating series would be:

$$\begin{aligned} \tilde{F}_p(i) &= \tilde{F}_p(j) = \tilde{F}_p(k) \\ &= (-1)^N \tilde{F}_n(l) = (-1)^N \tilde{F}_n(m) = (-1)^N \tilde{F}_n(n). \end{aligned} \tag{A.12}$$

One of us has obtained a relation similar to (A.12) (cf equation (25) in Raynal 1978). But that relation is different since it is valid for the 3-j coefficient when expressed in terms of a scaled ${}_3F_2$.

Of the three generators g_1, g_2, g_3 for G_{T}, in the text, for the generator g_1, the 5×5 matrix representating the Weber–Erdelyi transformation (6), denoted by (I) above, was chosen. The 72 elements of the 5×5 representation for G_{T} can also be generated if g_1 is anyone of the matrices representing the transformation (V)–(X) or (XVIII). However, if for g_1, the 5×5 unit matrix representing (XII) is chosen, then it would result in a four-element subgroup of G_{T}. Similarly, choosing (XI) for g_1 results in an eight-element subgroup of G_{T}; choosing (II), (III), (XIV) or (XV) for g_1 results in 12-element subgroups of G_{T}; and choosing (IV), (XIII), (XVI) or (XVII) results in 36-element subgroups of the group G_{T}.

When $c = \alpha_{345} = -N$ determines the termination of the ${}_3F_2$ series, from the definition (3) for F_p, it follows that (m, n) can take only the three values (3,4), (3,5) or (4,5). Since any one of the numerator parameters of $F_p(l)$ (namely, $\alpha_{imn}, \alpha_{jmn}, \alpha_{kmn}$) can be α_{345}, the indices i, j, k are restricted to 5, 4 or 3, which in turn implies that l can be only 0, 1 or 2. Therefore, (m, n) being any two of 3, 4, 5 ($^3C_2$) and l being any one of 0, 1, 2 ($^3C_1$), it is obvious that α_{345} can occur as a numerator parameter in only ($^3C_1 \times ^3C_2 =$) nine series. When r_i is replaced by $-r_i$, instead of the $F_p(l)$ series, the $F_n(l)$ series arise. From the definition (4) for the $F_n(l)$ series, (j,k), (i,k) or (i,j) can take the values (3,4), (3,5) or (4,5) so that l can be 5, 4 or 3 ($^3C_1$) and (m,n) can be only (0,1), (0,2) or (1,2). Once again there are only nine F_n series. This explains why in the relations (A.1) to (A.5) amongst the 18 terminating ${}_3F_2$ series, $F_p(0)$, $F_p(1)$, $F_p(2)$ and $F_n(3)$, $F_n(4)$, $F_n(5)$ alone occur.

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